The Influence Of Curved Beams In The Static And Dynamic Finite Element Analysis Of Lighting Columns

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Abstract

Curved beams have found many applications in various fields of engineering such as civil, mechanical and aerospace engineering. Therefore, many researchers have devoted themselves to developing finite element solutions for curved beams. The question of the choice of the proper finite element solution to perform static and dynamic analysis of the structures that involve curved beams was investigated. The investigation was carried out on a lighting column. Two finite element solutions named; straight beam finite element method (SE-method) and curved beam finite element method (CE-method) were studied. The results for natural frequencies, mode shapes, and the deformed configurations obtained from the two approaches were compared. A MATLAB code was written to carry out the static and dynamic finite element analysis of the lighting column.

Keywords: Curved beams, finite element method, static and dynamic analysis, lighting column.

1. Introduction

The finite element analysis of curved beam has been a topic of intense interest for researchers over years. The motivation for this activity is largely because; **i**) curved beams have many applications in engineering such as civil, mechanical and aerospace engineering. **ii**) Curved beams are more efficient than straight beams in transferring loads because the transfer is influenced by shear, bending and membrane action [1 ,2]. **iii**) Shear and membrane locking problems are one of the most serious difficulties that challenge researchers to obtain the exact stiffness and mass matrices for curved beam elements. This is because locking phenomena leads to underestimation of the bending deformations and overestimations of natural frequency for curved beams [3].

For most structural engineers, choosing the suitable finite element method for analysing curved structures, such as rings, arches and bridges, is still very difficult and tricky [4 ,5]. Generally, two methods namely SE-method and CE-method can be employed for analysing curved beams/arches, with SE-method referring to straight finite beam elements and CE to curved finite beam elements. The SE-method is based on the conventional straight beam elements, where a curved beam is idealized as a sequence of a series of small straight beam segments in order to approximate the true curved shape. The other approach is to utilise the curved beam finite element solution to analyse curved structures.

The main purpose of this study is to compare the SE-method with the CE-method to analyse a structure that has a curved beam. The static and dynamic comparison was carried out on a street lighting column structure. This comparison was included natural frequencies, mode shapes and deformed configurations.

2. Description of lighting columns

The lighting column, shown in Figure 1, was chosen in this study. It consists of three parts, two straight parts and a curved part which is a quarter of circle. The mounting height of the pole is 13.2 m, see Figure 1 for additional dimensions. The cross section of the lighting column is a hollow circular section. A lamp, which is not shown in the Figure 1, is attached to the tip of the lighting column. The lamp has a mass of 40 kg. The required material properties for the lighting column are assumed to be; Modulus of elasticity *E=200 GPa* and mass density *ρ= 7850 kg/m3*.

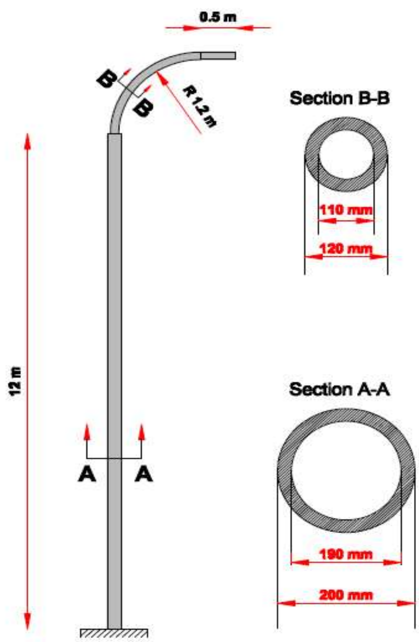

Figure 1: Detailed dimension of the lighting column.

3. Finite element modelling of the lighting columns

In this paper, two finite element solutions were conducted on the lighting column shown in Figure 1. The first is the SE method that based on the conventional straight-beam elements, in which the three parts of the lighting column were modelled by straight beam elements. In the conventional straight beam elements, a curved beam is divided into a series of straight beam segments to approximate the true curved shape. The other FE solution is the CE method which is based on curved beam elements where the curved part of the lighting column was modelled by curved beam elements and the other parts by straight beam elements. A comparison between the two finite element solutions was made in terms of displacements, natural frequencies and mode shapes. A MATLAB code was written for the purpose of this research.

It is well know that the accuracy of the static and dynamic analyses by using FEM is dependent on stiffness and mass matrices. Thus, stiffness and mass matrices for both straight beam finite element and curved beam finite element are presented below.

3.1. Finite element formulation of straight beam element

Figure 2 shows a two dimensional straight beam element with uniform cross section and it has six degrees of freedom, three at each node. The beam is capable of resisting axial forces (F_{X1}, F_{X2}), shearing forces (F_{Y1}, F_{Y2}) and bending moments (M_1, M_2). The nodes are labelled using numbers inside circles. The undeformed beam is shown with a solid line, while the deformed beam is shown with a dash line. The capital letters are used to correspond to the local coordinate, while small letters are used to represent the global coordinates [6].

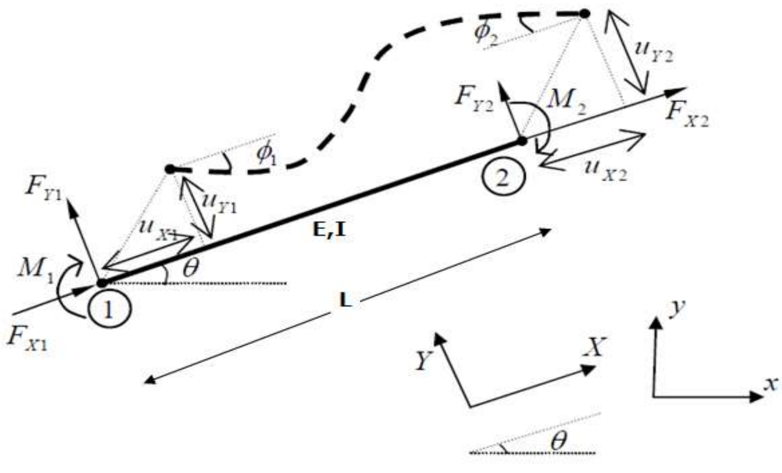

Figure 2: Straight beam element.

Where (L) is length of the beam, (I) is moment of inertia of the cross-sectional area, (E) is Elastic Modulus and (u_X and u_Y) are corresponding displacements.

The stiffness matrix in local coordinate system [KL] for the plane straight beam element shown in Figure 2 is given by equation (1);

$$KL=\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^2} & \frac{-6EI}{L^2} & 0 & \frac{-12EA}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4E}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (1)$$

To transform the local stiffness matrix from local coordinates to global coordinates, the following relationship is used:

$$[KG]= [T]^T [KL] [T] \quad (2)$$

Where [KG] is the global stiffness matrix and [T] is the transformation matrix, which can be written as:

$$[T]=\begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Where $c = \cos \Theta$ *and* $s = \sin \Theta$

The elemental mass matrix in local coordinate system [ML] for the finite straight plane beam element shown in Figure 2 can be computed as follows;

$$ML = \frac{\rho A}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & -22L & 0 & 54 & 13L \\ 0 & -22L & 4L^2 & 0 & -13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & -13L & 0 & 156 & 22L \\ 0 & 13L & -3L^2 & 0 & 22L & 4L^2 \end{bmatrix} \quad (4)$$

Where ρ is the mass density of the material and A is the cross-sectional area. Similarly, the global mass matrix [MG] can be obtained from the following equation:

$$[MG]= [T]^T [ML] [T] \quad (5)$$

3.2. Finite element formulation of curved beam element

Many formulations have been derived for a curved element [1,2,3,4,5 and 7]. These formulations are very complex which discourage designers from employing them. Stiffness and mass matrices derived by [4] for a curved beam finite element are used in the present study, since they presented the most explicit forms of the curved element property matrices.

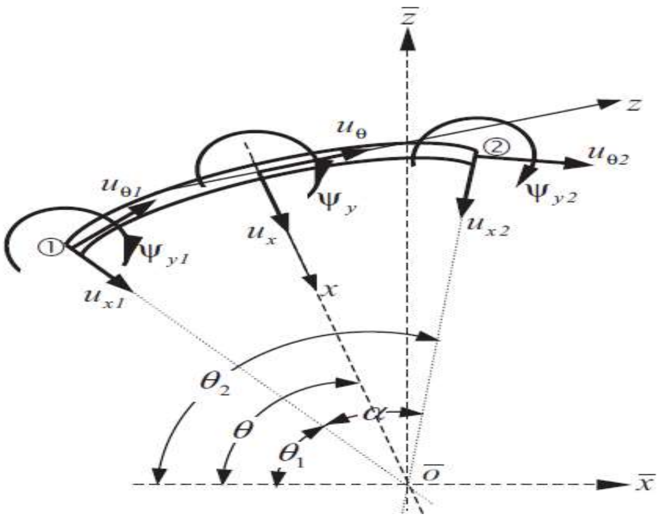

Figure 3: Curved beam element [4].

Figure 3 shows a three dimensional curved beam element, where ux is the radial displacement, u_θ is the circumferential displacement and φ_y is the rational angle.

The stiffness matrix of the curved element, shown in Figure 3, in local coordinate system [KL] obtained from the following relation:

$$[KL] = [D]\,[B]^{-1} \tag{6}$$

Where:

$$\mathbf{D} = \frac{EI_y}{R^2}\begin{bmatrix} 0 & 0 & 0 & -\frac{2}{R}\sin\theta_1 & 0 & -\frac{2}{R}\cos\theta_1 \\ 0 & 0 & 0 & \frac{2}{R}\cos\theta_1 & 0 & -\frac{2}{R}\sin\theta_1 \\ -1 & 0 & 0 & -2\cos\theta_1 & 0 & -2\sin\theta_1 \\ 0 & 0 & 0 & \frac{2}{R}\sin\theta_2 & 0 & \frac{2}{R}\cos\theta_2 \\ 0 & 0 & 0 & -\frac{2}{R}\cos\theta_2 & 0 & \frac{2}{R}\sin\theta_2 \\ 1 & 0 & 0 & 2\cos\theta_2 & 0 & -2\sin\theta_2 \end{bmatrix} \tag{7}$$

$$C = 1 + \left(\frac{I_y}{AR^2}\right) \tag{8}$$

$$I_y = \int_A x^2/(1 - \frac{X}{R})\ dA \tag{9}$$

Where R is the average radius of the arch curvature and I_y is the moment of interia of the area A about the y-axis.

$$B = \begin{bmatrix} 1 & \cos\theta_1 & \sin\theta_1 & \theta_1\sin\theta_1 & 0 & \theta_1\cos\theta_1 \\ 0 & \sin\theta_1 & -\cos\theta_1 & \sin\theta_1 - \theta_1\cos\theta_1 & 1 & \cos\theta_1 + \theta_1\sin\theta_1 \\ \frac{C}{R}\theta_1 & 0 & 0 & \frac{2}{R}\sin\theta_1 & \frac{1}{R} & \frac{2}{R}\cos\theta_1 \\ 1 & \cos\theta_2 & \sin\theta_2 & \theta_2\sin\theta_2 & 0 & \theta_2\cos\theta_2 \\ 0 & \sin\theta_2 & -\cos\theta_2 & \sin\theta_2 - \theta_2\cos\theta_2 & 1 & \cos\theta_2 + \theta_2\sin\theta_2 \\ \frac{C}{R}\theta_2 & 0 & 0 & \theta_2\sin\theta_2 & \frac{1}{R} & \frac{2}{R}\cos\theta_2 \end{bmatrix} \tag{10}$$

In order to transform the stiffness and mass matrices of each curved beam element from the local coordinate system to the global coordinate before they are assembled, the following equation is used:

$$[KG]= [T]^{T} [KL] [T] \tag{11}$$

$$T = \begin{bmatrix} c1 & s1 & 0 & 0 & 0 & 0 \\ -s1 & c1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c2 & s2 & 0 \\ 0 & 0 & 0 & -s2 & c2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Where is C1=cos$_{\Theta 1}$, S1= cos$_{\Theta 1}$, C2= cos$_{\Theta 2}$ and S2= cos$_{\Theta 2}$.

The elemental mass matrix in local coordinate system [ML] for the curved beam element shown in Figure 3 is giveb by;

$$[ML]= \rho R\, [B^{-1}]^{T} [\overline{H}]\, [B]^{-1} \tag{13}$$

It should be noted that, $[\overline{H}]$ is a 6*6 a symmetrical square matrix, and in order to obtain the global mass matrix equation 5 should be used. Also, the nodal displacements are ordered as shear, axial and moment respectively.

$$\overline{H} = \begin{bmatrix} \overline{H}_{11} & \overline{H}_{12} & \overline{H}_{13} & \overline{H}_{14} & \overline{H}_{15} & \overline{H}_{16} \\ \overline{H}_{21} & \overline{H}_{22} & \overline{H}_{23} & \overline{H}_{24} & \overline{H}_{25} & \overline{H}_{26} \\ \overline{H}_{31} & \overline{H}_{32} & \overline{H}_{33} & \overline{H}_{34} & \overline{H}_{35} & \overline{H}_{36} \\ \overline{H}_{41} & \overline{H}_{42} & \overline{H}_{43} & \overline{H}_{44} & \overline{H}_{45} & \overline{H}_{46} \\ \overline{H}_{51} & \overline{H}_{52} & \overline{H}_{53} & \overline{H}_{54} & \overline{H}_{55} & \overline{H}_{56} \\ \overline{H}_{61} & \overline{H}_{62} & \overline{H}_{63} & \overline{H}_{64} & \overline{H}_{65} & \overline{H}_{66} \end{bmatrix} \tag{14}$$

The coefficients of matix $[\overline{\boldsymbol{H}}]$ is given in Appendix A.

4. Numerical results and discussions

4.1. Static analysis of the lighting column

Figures 4 depict the deformed shape of the lighting column that was obtained from the SE and CE finite element methods. The undeformed shape of the lighting column is shown with a solid line, while the deformed shape is shown with a dash line. It is worthwhile to mention that in the SE-method a large number of straight beam elements were used to model the curved part of the lighting column in order to converge towards the CE-method solution.

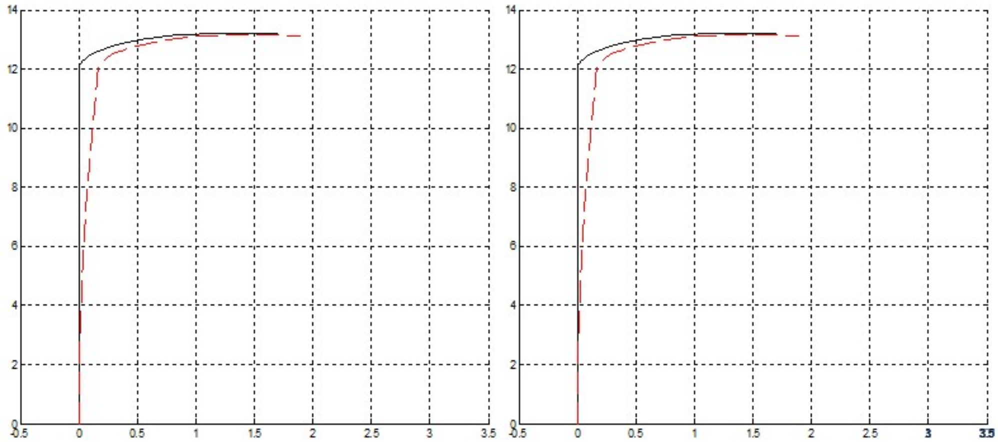

Figure 4: Deformed shape of modelling the lighting column by SE-method (left) and CE-method (right).

4.2. Dynamic analysis of the lighting column

Table (1) shows the lowest six natural frequencies of the lighting column that obtained from SE and CE finite element methods. From table (1) it is evident that, all the natural frequencies obtained from the SE-method are similar to those of the CE-method for the lighting column.

Table 1: The lowest six natural frequencies of the lighting column.

Mode shape number	Natural frequencies (ω rad/sec)	
	SE-method	CE-method
1	1.07	1.07
2	6.50	6.51
3	16.47	16.47
4	26.93	26.94
5	43.54	43.54
6	67.98	67.98

Figures 5 (a), (b), (c) and (d) shows the lowest four mode shapes of the lighting column from the SE and CE finite element methods respectively. It is seen that, the mode shapes obtained from the both methods are exactly the same.

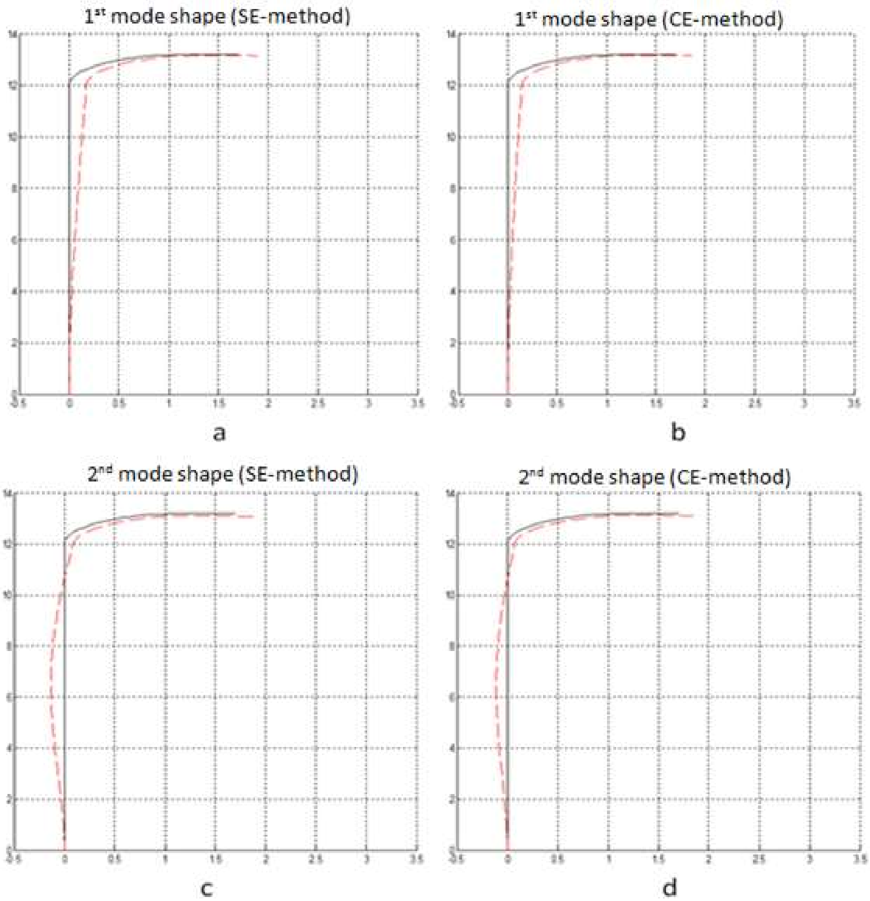

Figure (5): Comparison of the first and the second mode shapes of the lighting column from the SE and CE finite element methods.

5. Conclusion

The results for natural frequencies, mode shapes, and the deformed configurations of a lighting column which were obtained from the SE and CE methods were compared. Conclusions drawn from this comparison are as follows:

- Using either the SE Finite element method or the CE finite element method, the natural frequencies and the associated mode shapes of the lighting column obtained from both methods are almost the same. Thus, for simplicity, one may use the simple mass matrix of straight beam element (SE-method) instead of the curved beam element mass matrix (CE-method) to perform the dynamic analysis of the lighting column.

- The accuracy of displacements of the lighting column obtained by using the SE-method may be as accurate as that obtained by using the CE-method, if the number of straight beam elements is large enough.
- Increase in the total number of elements in CE finite element method hardly affects the results. However, large number of elements should be used in the SE finite element method to converge towards the CE-method result.

I recommend for future work to investigating the effect of the radius of curvature on displacements obtained by SE and CE approaches.

6. References

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Appendix A: coefficients of matrix $[\overline{H}]$

$$\overline{H}_{11} = [A\,\theta \;+\; c^2\;\left(A + \frac{I_y}{R^2}\right)\left(\frac{\theta^3}{3}\right)]_{\theta_1}^{\theta_2}$$

$$\overline{H}_{21} = A[(1 + C)\,\sin\theta - C\theta\,\cos\theta\,]_{\theta_1}^{\theta_2}$$

$$\overline{H}_{22} = A[\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{31} = A[-(1+C)\cos\theta\sin\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{32} = 0$$

$$\bar{H}_{33} = A[\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{41} = [(A + 3AC + \frac{2cI_y}{R^2})\,(\,\sin\theta - \theta\cos\theta) - AC\theta^2\sin\theta\,\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{42} = \frac{A}{2}[\theta\,-\,\sin\theta\cos\theta\,]_{\theta_1}^{\theta^2}$$

$$\bar{H}_{43} = \frac{A}{2}\,[\theta^2\,-\,\sin^2\theta]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{44} = \left[\frac{A\theta^3}{3}\,+\,(A\,+\,\frac{2I_y}{R^2})\,(\,\theta - \sin\theta\cos\theta) - A\theta\,\,\sin^2\theta\,\,\right]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{51} = \left[(\,\frac{C}{2}\,) + \,(A\,+\,\frac{I_y}{R^2}\,)\theta^2\,\,\right]_{\theta_1}^{\theta^2}$$

$$\bar{H}_{52} = A\,[-\cos\theta\,\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{53} = A\,[-\sin\theta\,\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{54} = [-\,(2A\,+\,\frac{2\mathrm{I}_{\mathrm{y}}}{R^2}\,)\cos\theta - A\theta\sin\theta\,\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{55} = [(\,A + \frac{I_y}{R^2})\,\theta\,]_{\theta_2}^{\theta_2}$$

$$\bar{H}_{61} = [\,(A\,+\,3AC + \frac{2CI_y}{R^2}\,)(\cos\theta +\,\sin\theta) - A\,C\theta^2\cos\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{62} = \frac{A}{2}\,\,[\theta^2 +\,\sin^2\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{63} = \frac{A}{2}\,[\,-\theta - \sin\theta\cos\theta]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{64} = \frac{1}{2}\,[-\,A(\theta\sin 2\theta + \frac{\cos 2\theta}{2}\,)\,\,+\,(A\,+\frac{4I_y}{R^2}\,)\sin^2\theta]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{65} = [(\,2A + \frac{2\,I_y}{R^2})\sin\theta - A\theta\cos\theta\,]_{\theta_1}^{\theta_2}$$

$$\bar{H}_{66} = [\frac{A\theta^3}{3} +\,A\sin\theta(\cos\theta + \theta\sin\theta) + (\frac{2I_y}{R^2}\,)\,(\theta +\,\sin\theta\cos\theta)]_{\theta_1}^{\theta_2}$$